BAB VIII

FUNGSI GAMMA DAN FUNGSI BETA

Tujuan Pembelajaran

Fungsi gamma dan beta merupakan fungsi-fungsi istimewa yang sering muncul dalam pemecahan persamaan differensial, proses fisika, perpindahan panas, gesekan sumber bunyi, rambatan gelombang, potensial gaya, persamaan gelombang, mekanika kuantum dan lainnya. Fungsi gamma dan beta merupakan fungsi dalam bentuk pernyataan integral dan mudah untuk dipelajari. Kedua fungsi ini biasanya dibahas secara rinci dalam fungsi bilangan kompleks. Disini hanya dibahas secara definisi dan sifat-sifat sederhana yang dimiliki fungsi tersebut.

A. Fungsi Gamma

Fungsi gamma didefinisikan dalam bentuk:

$$\Gamma(n) = \int_0^\infty x^{n-1}.e^{-x}\,dx$$

yang konvergen untuk n > 0.

Hasil integral menyatakan bahwa $\Gamma(n) = (n-1)\,!$. Secara recrussive fungsi gamma dapat ditulis $\Gamma(n) = (n-1) = n\,!$

$\Gamma(n)$ disebut juga fungsi factorial, yaitu perkalian berlanjut, dimana n = 1, 2, 3, …

Contoh:

$$\Gamma(6) = 5! = 1.2.3.4.5 = 120$$

$$\frac{\Gamma(5)}{\Gamma(3) =} = \frac{4!}{2!} = \frac{24}{2} = 12$$

Bentuk khusus untuk $\Gamma(\tfrac{1}{2}) = \sqrt{\pi}$ (Lihat pembuktian). Bentuk $\Gamma(n)$ di atas, berlaku untuk n > 0. Bila n < 0, bentuk diatas dapat ditulis:

$$\Gamma(n) = \frac{\Gamma(n+1)}{n}$$

Contoh :

$$\Gamma(-\tfrac{1}{2}) = \frac{\Gamma(-\tfrac{1}{2}+1)}{-\tfrac{1}{2}} = \frac{\Gamma(-\tfrac{1}{2})}{-\tfrac{1}{2}} = -2\sqrt{\pi}$$

$$\Gamma\left(-\tfrac{3}{2}\right) = \frac{\Gamma(-\tfrac{3}{2}+1)}{-\tfrac{5}{2}} = \frac{\Gamma(-\tfrac{1}{2})}{-\tfrac{3}{2}} = \frac{\Gamma(-\tfrac{1}{2}+1)}{-\tfrac{3}{2}.-\tfrac{1}{2}} = \frac{\Gamma(\tfrac{1}{2})}{-\tfrac{3}{2}.-\tfrac{1}{2}} = \tfrac{4}{3}\sqrt{\pi}$$

$$\Gamma\left(-\tfrac{5}{2}\right) = \frac{\Gamma(-\tfrac{3}{2})}{-\tfrac{5}{2}} = \frac{\Gamma(-\tfrac{1}{2})}{-\tfrac{5}{2}.-\tfrac{3}{2}} = \frac{\Gamma(\tfrac{1}{2})}{-\tfrac{5}{2}.-\tfrac{3}{2}.-\tfrac{1}{2}} = -\tfrac{8}{15}\sqrt{\pi}$$

Bukti:

$$\Gamma(n+1) = \int_0^\infty x^n.e^{-x}\,dx = \lim_{M\to\infty}\int_0^M x^n.e^{-x}\,dx$$

$$= \lim_{M\to\infty}\left[-x^n.e^{-x}\Big|_0^M - n\int_0^M x^{n-1}.e^{-x}\,dx\right]$$

$$= \lim_{M\to\infty}\left[0 + n\int_0^M x^{n-1}.e^{-x}\,dx\right]$$

$$= n\Gamma\Gamma(n$$

$$\text{jadi}: \Gamma(n+1) = n\,\Gamma(n) = n\,!$$

Contoh:

1. $\displaystyle\int_0^\infty x^3.e^{-x}\,dx = \Gamma(4) = 3! = 6$

2. $\displaystyle\int_0^\infty x^6.e^{-2x}\,dx = \int_0^\infty \left(\tfrac{u}{2}\right)^6 e^{-u}\,\frac{du}{2} = \tfrac{1}{2^7}\int_0^\infty u^6.e^{-u}\,du$

$$\frac{\pi(7)}{2^7} = \frac{1.2.3.4.5.6}{2.2.2.2.2.2.2} = \frac{45}{8}$$

3. $\displaystyle\Gamma(\tfrac{1}{2}) = \int_0^\infty x^{-\frac{1}{2}}.e^{-x}\,dx$, misalkan $x = u^2$

$$= 2\int_0^\infty e^{-u^2}\,du$$

$$\left[\Gamma(\tfrac{1}{2})\right]^2 = \left\{2\int_0^\infty e^{-u^2}\,du\right\}\left\{2\int_0^\infty e^{-v^2}\,dv\right\}$$

$$= 4\int_0^\infty\int_0^\infty e^{-(u^2+v^2)}\,du.dv$$

transformasikan ke koordinat poolar (ρ,θ) dengan $u=\rho\cos\theta$ dan $v=\rho\sin\theta$; du $dv=\rho\, dp.d\theta$.

$$\left[\Gamma\left(\tfrac{1}{2}\right)\right]^2 = 4\int_{\theta=0}^{\pi/2}\int_{\rho=0}^{\sigma/2} e^{-\rho}.\rho\rho d\rho\rho d\theta = 4\int_0^{\pi/2}\left[-\tfrac{1}{2}.e^{-\rho^2}\right]_0^{\infty} d\theta$$

$$= 4.\tfrac{1}{2}\int_0^{\pi/2} d\theta = 2\theta\Big|_0^{\pi/2} = \pi$$

$$\text{Jadi}: \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi} = 1{,}772454 \rightarrow \pi = 3{,}141593$$

Contoh dalam penyelesaian PD.

Sebuah partikel yang berada di ujung pegas, ditarik menjauhi titik awal O oleh gaya yang berbanding terbalik dengan kecepatan sesaatnya dari titik O. Bila partikel itu dilepas, tentukan waktu untuk sampai ke titik O lagi.

Jawab :

Pada saat t = 0, misalkan partikel ditarik sejauh a pada sumbu x, pada x = a, dari titik asal x = 0.

Menurut Hukum Newton

$$m.\frac{d_2x}{dt^2} = -\frac{k}{x}$$

dengan m = massa partikel dan k = konstanta pegas.

Misalkan $\dfrac{dx}{dt} = v$, kecepatan partikel pada tiap saat, maka:

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = \frac{dv}{dx}.\frac{dx}{dt} = v.\frac{dv}{dt} \text{ persamaanmenjadi}$$

$$m.v.\frac{dv}{dx} = -\frac{k}{x}$$

$$\text{atau}: \tfrac{1}{2}.m.v^2 = -k\ln x + \ln c$$

$$\tfrac{1}{2}.m.v^2 = \ln\frac{C}{x^k}$$

$$\text{Syaratbatas}:$$

$$\left.\begin{matrix} t=0 \\ x=a \end{matrix}\right\} 0 = \ln\frac{C}{x^k}$$

$$v = 0 \quad C = a^k$$

maka:

$$\tfrac{1}{2}mv^2 = \ln\left(\frac{a}{x}\right)^k$$

$$v^2 = \frac{2k}{m}\ln\frac{a}{x}$$

$$v = \frac{dx}{dt} = \sqrt{\frac{2k}{m}}.\sqrt{\ln\frac{a}{x}} \qquad \text{(persamaankecepatangerak)}$$

selanjutnya didapatkan waktu untuk sampai ke titik awal 0 selama (x = a ke x = 0)

$$t = \int_a^0 \sqrt{\frac{m}{2k}} \cdot \frac{dx}{\sqrt{\ln\frac{a}{x}}} = \sqrt{\frac{m}{2k}} \int_a^0 \left(\ln\frac{a}{x}\right)^{-\frac{1}{2}} dx$$

Misalkan:

$$\ln\frac{a}{x} = u \text{ atau } x = ae^{-u}, \text{persamaanmenjadi}:$$

$$t = a\sqrt{\frac{m}{2k}}\int_0^\infty u^{-\frac{1}{2}}.\ e^{-u}\,du = a\sqrt{\frac{m}{2k}}.\Gamma(\tfrac{1}{2}) = a\sqrt{\frac{m\pi}{2k}}$$

$$\text{Jadi waktu untuk ke titik 0 adalah } t = a\sqrt{\frac{m\pi}{2k}} \text{ detik}$$

Soal-soal :

1. Hitunglah!

   a. $\frac{\Gamma(6)}{\Gamma(3)} =$

   b. $\frac{\Gamma(\frac{5}{2})}{\Gamma(\frac{1}{2})} =$

   c. $\frac{\Gamma(3).\Gamma(2,5)}{\Gamma(5,5)} =$

   d. $\frac{6.\Gamma(\frac{8}{3})}{5.\Gamma(\frac{2}{3})} =$

   e. $\frac{2.\Gamma(8)}{\Gamma(4,5)} =$

   f. $\Gamma(-\frac{7}{2})$

   g. $\frac{\Gamma(-\frac{3}{2})}{\Gamma(\frac{3}{2})} =$

   h. $\Gamma(-\frac{3}{2}).\Gamma(\frac{3}{2}) =$

   i. $\frac{\Gamma(\frac{7}{2})}{\Gamma(-\frac{1}{2})} =$

   j. $\frac{\Gamma(-\frac{3}{2})}{\Gamma(-\frac{5}{2})} =$

2. Selesaikan!

a. $\int_0^\infty x^4 . e^{-2x}\, dx =$

b. $\int_0^\infty x^3 . e^{-\frac{1}{2}x}\, dx =$

c. $\int_0^\infty x^4 . e^{4x}\, dx =$

d. $\int_0^\infty x^2 . e^{-4x}\, dx =$

e. $\int_0^\infty x^{-\frac{1}{2}} . e^{-2x}\, dx =$

3. Selesaikan!

a. $\int_0^\infty \sqrt{x}\, .\, e^{-x^3}\, dx$

b. $\int_0^\infty 3^{-4x^2}\, dx$

c. $\int_0^1 \frac{dx}{\sqrt{-\ln x}} =$

d. $\int_0^1 x^m . e^{-ax^n}\, dx =$

B. Fungsi Beta

Fungsi beta dinyatakan dalam bentuk:

$$B(m,n) = \int_0^1 x^{m-1}\,(1-x)^{n-1}\, dx$$

yang konvergen untuk m > 0 ; n > 0.

Hubungan fungsi beta dengan gamma adalah

$$B(m,\ n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$

Banyak masalah integral yang dapat diselesaikan dalam bentuk fungsi beta atau gamma. Bentuk-bentuk umum yang dapat diselesaikan dengan fungsi beta adalah:

1. $\int_0^{\pi/2} \sin^{2m-1}\theta\ \cos^{2n-1}\theta\ d\theta = \frac{1}{2} B(m,n) = \frac{\Gamma(m).\Gamma(n)}{2\,\Gamma(m+n)}$

   valid untuk m > 0 dan n > 0

2. $\int_0^\infty \frac{x^{p-1}}{(1+x)} dx = \Gamma(p).\Gamma(1-p) = \frac{\pi}{\sin \rho\, \pi}$ untuk $0 < p < 1$

Contoh :

1. Buktikan B(m,n) = B(n,m), sifat simetris

   Bukti:

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \qquad \text{misalkan}: x = (1-y)$$

$$= \int_0^1 (1-y)^{m-1}.\,y^{n-1}\,dy$$

$$= \int_0^1 y^{n-1}.(1-y)^{m-1}\,dy$$

$$= B(n,m)$$

$$\text{Jadi}: B(m,n) = B(n,m)$$

1. Buktikan $B(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta.\cos^{2n-1}\theta\,d\theta$

   Jawab:

   Misalkan $x = \sin^2\theta \to x = 1,\ \theta = \pi/2$

$$B(m,n) = \int_0^1 x^{m-1}.(1-x)^{n-1}\,dx; \to x = 0,\ \theta = 0$$

$$B(m,n) = \int_0^{\pi/2} (\sin^2\theta)^{m-1}.\,(\cos^2\theta)^{n-1}.2\sin\theta\,\cos\theta\,d\theta$$

$$B(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta\,\cos^{2n-1}\theta\,d\theta \to (\text{terbukti})$$

2. Buktikan $B(m,\ n) = \dfrac{\Gamma(m)\,.\Gamma(n)}{\Gamma(m+n)}$, *untuk* $m > 0$ dan $n > 0$

   Jawab :

$$\text{Misal, } u = x^2 \text{ maka } \Gamma(m) = \int_0^\infty u^{m-1}.e^{-u}\,du$$

$$= 2\int_0^\infty x^{2m-1}.e^{-x^2}\,dx$$

   Analogi: $\Gamma(n) = 2\int_0^\infty y^{2n-1}.e^{-y^2}\,dy$

Jadi :

$$\Gamma(m).\Gamma(n) = 4\int_0^\infty\int_0^\infty x^{2m-1}\, y^{2n-1}\, e^{-(x^2+y^2)}\, dx\, dy$$

Transformasikan ke dalam koordinat poolar, didapat:

$$\Gamma(m).\Gamma(n) = 4\int_0^{\pi/2}\int_{\rho=0}^{\infty}\rho^{2(m+1)-1}.e^{-\rho^2}.\cos^{2m-1}\theta.\sin^{2n-1}\theta\, dp\, d\theta$$

$$= 4\left(\int_0^\infty \rho^{2(m-n)}.e^{-\rho 2}\, dp\right)\left(\int_0^{\pi/2}\cos^{2m-1}\theta.\ \sin^{2n-1}\theta\ d\theta\right)$$

$$= 2\,\Gamma(m+n).\int_0^{\pi/2}\cos^{2m-1}\theta.\ \sin^{2n-1}\theta\, d\theta$$

$$\Gamma(m).\Gamma(n) = \Gamma(m+n).B(m,n)$$

$$\text{Jadi}: B(m,n) = \frac{\Gamma(m).\Gamma(n)}{\Gamma(m+n)} \rightarrow \text{Terbukti}$$

Contoh Soal :

1. $\int_0^1 x^4(1-x)^3\, dx = B(5,4) = \frac{\Gamma(5).\Gamma(4)}{\Gamma(9)} = \frac{4!.3!}{8!} = \frac{1}{280}$

2. $\int_0^2 \frac{x^2\, dx}{\sqrt{2-x}} = 4\sqrt{2}\int_0^1 \frac{u^2}{\sqrt{1-u}}\, du, \quad misal: x = 2u; \quad dx = 2\,du$

$$= 4\sqrt{2}\int_0^1 u^2.(1-u)^{-\frac{1}{2}}\, du$$

$$= 4\sqrt{2}\, B(3, \tfrac{1}{2})$$

$$= 4\sqrt{2}.\frac{\Gamma(3).\Gamma(\frac{1}{2})}{\Gamma(3,\frac{1}{2})}$$

$$= \frac{4\sqrt{2}.2!\sqrt{\pi}}{\frac{5}{2}.\frac{3}{2}.\frac{1}{2}.\Gamma(\frac{1}{2})}$$

$$= \frac{64}{15}\sqrt{2} = 6{,}03398$$

3. Hitung $\int_0^{\infty} \sqrt{z}\,.e^{-z^3}\ dz$

Jawab :

Misalkan $Z = t^{-\frac{1}{3}} \rightarrow dz = \frac{1}{3}\,t^{-\frac{2}{3}dt}$

Jadi

$$I = \int_0^{\infty} \sqrt{t^{\frac{1}{3}}}\,.e^{-t}.\tfrac{1}{3}\,t^{-\frac{2}{3}}\ dt$$

$$= \tfrac{1}{3}\int_0^{\infty} t^{\frac{1}{6}-\frac{2}{3}}.e^{-t}.dt$$

$$= \tfrac{1}{3}\int_0^{\infty} t^{-\frac{1}{2}}\,.e^{-t}\ dt$$

$$= \tfrac{1}{3}\,\Gamma\left(\tfrac{1}{2}\right) = \tfrac{1}{3}\sqrt{\pi} = 0{,}59082$$

4. Bila $\int_0^{\infty} \frac{x^{p-1}}{1+x}dx = \frac{\pi}{\sin p\pi}$, buktikan bahwa $\Gamma(p).\Gamma(1-p) = \frac{\pi}{\sin p\pi}$

Jawab :

Ambil :

$$\frac{x}{1+x} = y \text{ atau } x = \frac{y}{1-y}$$

$$x = 0 \rightarrow y = 0$$

$$x = \infty \rightarrow y = 1$$

Integral menjadi: $\int_0^{1} y^{p-1}(1-y)^{-p}\ dy = B(p,1-p)$

$$= \frac{\Gamma(p).\Gamma(1-p)}{\Gamma(p+1-p)}$$

$$= \Gamma(p).\Gamma(1-p) \rightarrow \text{ Terbukti}$$

5. Selesaikan $\int_0^{\infty} \frac{dy}{1+y^4}$

Jawab :

Misalkan $y^4 = x \rightarrow 4y^3dy = dx \rightarrow dy = \frac{dx}{4x^{\frac{3}{4}}}$

Integral menjadi:

$$= \tfrac{1}{4}\int_0^\infty \frac{x^{-\frac{3}{4}}}{1+x}\,dx = \tfrac{1}{4}.B(\tfrac{1}{4},1-\tfrac{1}{4})$$

$$= \tfrac{1}{4}\int_0^\infty x^{-\frac{1}{4}}(1+x)^{-1}\,dx$$

$$= \tfrac{1}{4}.B(\tfrac{1}{4},1-\tfrac{1}{4})$$

$$= \tfrac{1}{4}.\Gamma(\tfrac{1}{4}).\Gamma(1-\tfrac{1}{4})$$

$$= \tfrac{1}{4}.\frac{\Gamma}{\sin\frac{\pi}{4}}$$

$$= \frac{\pi}{4}\sqrt{2}$$

6. Selesaikan $\int_0^2 x\,\sqrt[3]{8-x^3}\,dx$

Jawab :

Misalkan

$$x^3 = 8y \rightarrow 3x^2\,dx = 8\,dy \rightarrow dx = \frac{8\,dy}{6\,y^{2/3}}$$

$$x = 2y^{\frac{1}{3}}$$

$$x = 2 \rightarrow y = 0$$

$$x = 0 \rightarrow y = 2$$

Integral menjadi:

$$\int_0^1 2y^{\frac{1}{3}}\,\sqrt{8(1-y)}.\tfrac{2}{3}\,y^{-\frac{2}{3}}\,dy$$

$$= \tfrac{8}{3}\int_0^1 y^{-\frac{1}{3}}(1-y)^{\frac{1}{3}}\,dy$$

$$= \tfrac{8}{3}\,B(\tfrac{2}{3},\tfrac{4}{3})$$

$$= \tfrac{8}{3}\,\frac{\Gamma(\frac{2}{3}).\Gamma(\frac{4}{3})}{\Gamma(2)}$$

$$= \tfrac{8}{3}\,\frac{\Gamma(\frac{2}{3}).\Gamma(\frac{1}{3}+1)}{1.1}$$

$$= \tfrac{8}{3}.\tfrac{1}{3}.\Gamma(\tfrac{2}{3}).\Gamma(\tfrac{1}{3}) = \tfrac{8}{9}\,\frac{\pi}{\sin\frac{\pi}{3}} = \frac{16\pi}{9\sqrt{3}}$$

$$= 3{,}224$$

Soal-soal :

1. Hitunglah!

   a. $\int_0^1 x^2(1-x)\,dx =$

   b. $\int_0^1 \sqrt{\frac{(1-x)}{x}}\,dx = e\int_0^2 (4-x^2)^{\frac{3}{2}}\,dx$

2. Hitunglah!

   a. $\int_0^4 u^{\frac{3}{2}}\,.(4-u)^{\frac{5}{2}}\,du =$

   b. $\int_0^3 \frac{dx}{\sqrt{3x-x^2}} =$

3. Hitunglah!

   a.

   $\int_0^{\pi/2} \sin^4\theta.\cos^4\theta\;d\theta =$

   b. $\int_0^{\pi/2} \cos^6\theta\;d\theta =$

   c. $\int_0^{\pi} \sin^5\theta\;d\theta =$

   d. $\int_0^{\pi/2} \cos^5\theta.\sin^2\theta\;d\theta =$

   e. $\int_0^{\pi/2} \sqrt{\tan\theta}\;d\theta =$

   f. $\int_0^{\pi} \sqrt{\cot\theta}\;d\theta =$

4. Buktikan bahwa:

a. $\displaystyle\int_0^{\infty} \frac{x\,dx}{1+x^6} = \frac{\pi}{3\sqrt{3}}$

b. $\displaystyle\int_0^{\infty} \frac{y^2\,dy}{1+y^4} = \frac{\pi}{2\sqrt{2}}$

c. $\displaystyle\int_{-\infty}^{\infty} \frac{e^{2x}}{(e^{3x}+1)^2}\,dx = \frac{2\pi}{9\sqrt{3}}$

5. Buktikan bahwa: $\displaystyle\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \,.\cos^{2n-1}\theta\; d\theta = \frac{\Gamma(m)\,.\Gamma(n)}{2\,\Gamma(m+n)}; m, n > 0$